

Residues : The gateway to higher arithmetic I

Recalling section one of Gauss's *Disquisitiones Arithmeticae*

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ABSTRACT: Residues to a given modulus have been introduced to mathematics by Carl Friedrich Gauss with the definition of congruence in the ‘*Disquisitiones Arithmeticae*’. Their extraordinary properties provide the basis for a change of paradigm in arithmetic. By restricting residues to remainders left over by division Peter Gustav Lejeune Dirichlet — Gauss’s successor in Göttingen — eliminated in his ‘*Lectures on number theory*’ the fertile concept of residues and attributed with the number-theoretic approach to residues for more than one and a half centuries to obscure Gauss’s paradigm shift in mathematics from elementary to higher arithmetic.

1 The outset of Dirichlet’s number theory in a nutshell

The *Lectures on number theory* — posthumously published 1863 by Richard Dedekind — open as follows: *In this section we treat a few arithmetic theorems, which indeed may be found in most text books, but which are of such fundamental importance for our science that a rigorous proof is necessary.*¹ Section 1 starts with the proof of following general theorem extending the range of familiar elementary arithmetic:²

When one forms the product of any two out of three numbers, and then multiplies this product by the third number, the resulting product always has the same value, regardless of which two numbers are chosen first.

Since the product is independent of the order of the successive multiplications, it is called the product of the three numbers, and the latter are called its factors, without regard to order.

Subsequently it is shown that a similar theorem holds for any system S of arbitrary positive integers a, b, c, \dots :

The simplest way a product can be formed from these numbers is the following. One takes two numbers from S at random and forms their product; the System S' of the latter number and the remaining numbers then has one less number than S . By again choosing two numbers at random from S' and forming their product, one obtains a system S'' with the two numbers fewer than S . Continuing in this way, one finally arrives at a single number, and the theorem to be proved

¹Lejeune Dirichlet (1999, 1).

²Lejeune Dirichlet (1999, 2).

is that *the number remaining at the end of this process is always the same, no matter how the individual multiplications are ordered.*

To prove this we apply complete induction.

The last paragraph of the introductory chapter to arithmetic captioned with — *Looking back* — closes with an authoritative statement:³

Here we conclude the series of theorems on the divisibility of numbers. But it is worthwhile at this stage to look back along the path these investigations have taken. It is now clear that the whole structure rests on a single foundation, namely the algorithm for finding the greatest common divisor of two numbers. All the subsequent theorems, even when they depend on the later concepts of relative and absolute prime numbers, are still only simple consequences of the this initial investigation, so one is entitled to make the following claim: any analogous theory, for which there is a similar algorithm for the greatest common divisor, must also have consequences analogous to those in our theory. In fact, such theories exist.

§. 17 of chapter 2 — *On the congruence of numbers* — opens with a first consequence based on the initial investigation:⁴

If k is any positive integer, then each integer a , may be written in exactly one way in the form

$$a = sk + r,$$

where s is an integer and r is one of the k numbers

$$0, 1, 2, \dots, (k - 1).$$

⋮

In what follows we shall say that r is the *remainder* of the number a relative to the *modulus* k .

§. 18 then defines and treats to some extent *complete residue systems* comprising theorem 6 *of particular importance in later investigations*:⁵

If a is relatively prime to the modulus k and if one replaces x in the linear expression $ax + b$ by the series of all k terms in a complete system of incongruent numbers, then the resulting values also form a complete system of incongruent numbers.

³Lejeune Dirichlet (1999, 20).

⁴Lejeune Dirichlet (1999, 21).

⁵Lejeune Dirichlet (1999, 24).

The *remainder* of a number relative to the *modulus* k is used in §. 19 for the *proof of the generalised Fermat theorem* in the form $a^{\varphi(k)} \equiv 1 \pmod{k}$ which may be *expressed in words as follows*.⁶

If a is relatively prime to the positive integer k , and if one raises a to the power $\varphi(k)$, equal to the number of numbers among

$$1, 2, 3, \dots, k$$

relatively prime to k , then the result always leaves remainder 1 on division by k .

Afterwards, in §. 20 *another proof of the same theorem* is added with the accentuation: *It is not at all superfluous to give another proof of this important theorem which can be developed from the binomial theorem*.⁷

The explicit reference to *the binomial theorem* calls to mind an annotation to Fermat's theorem disclosing in section 3 of the *Disquisitiones Arithmeticae* a surprising detail to the theory of numbers.⁸

This theorem is worthy of attention both because of its elegance and its great usefulness. It is usually called Fermat's theorem after its discoverer. (See Fermat, *Opera Mathem.* p. 131) Euler first published a proof in his dissertation entitled "Theorematum quorundam ad numeros primos spectantium demonstratio," (Comm. acad. Petrop., 8 [1736], 1741, 141).^{*} The proof is based on the expansion of $(a+1)^p$. From the form of the coefficients it is easily seen that $(a+1)^p - a^p - 1$ will always be divisible by p and consequently so will $(a+1)^p - (a+1)$ whenever $a^p - a$ is divisible by p . Now since $1^p - 1$ is always divisible by p so also will $2^p - 2$, $3^p - 3$, and in general $a^p - a$. And since p does not divide a , $a^p - 1$ will be divisible by p . — *That will suffice to to clarify the character of that method*.⁹ The illustrious Lambert gave

⁶Lejeune Dirichlet (1999, 25–26).

⁷Lejeune Dirichlet (1999, 27).

⁸Gauss (1986, 32).

^{*}Gauss's footnote: In a previous commentary (Comm. Acad. Petrop., 6 1723–33], 1738, 106 ["Observationes de theoremate quodam Fermatiano aliisque ad numeros primos spectantibus"]), this great man had not yet reached this result. In the famous controversy between Maupertius and König on the principle of the least action, a controversy that led to strange digressions, König claimed to have at hand a manuscript of Leibniz containing a demonstration of this theorem very like Euler's (König, *Appel au public*, p. 106). We do not wish to deny this testimony, but certainly Leibniz never published his discovery. Cf. *Hist. de l'Ac. de Prusse*, 6 [1750], 1752, 530. [The reference is "Lettre de M. Euler á M. Merian (traduit du Latin)" which was sent from Berlin Sept. 3, 1752.]

⁹The reference to the character of the two different methods of proof of Fermat's theorem indicating for the first one the involvement of the induction principle has been omitted in the English translation. Because of its tremendous importance for the underlying principles of arithmetic an English translation was inserted on the basis of the Latin original.

a similar demonstration in *Nova Acta erudit.*, 1769, p. 109. But since the expansion of a binomial power seemed quite alien to the theory of numbers, Euler gave another demonstration that appears in *Novi comm. acad. Petrop.* T. VIII p. 70, which is more in harmony with what we have done in the preceding article. We will offer still others later. Here we will add another deduction similar to those of Euler . . .

The pointer to the preceding article which is said to be *more in harmony with what we have done* leads to take note of article 49:¹⁰

49. THEOREM. *If p is a prime that does not divide a , and a^t is the lowest power of a that is congruent to unity relative to the modulus p , the exponent t will be either $= p - 1$ or be a factor of this number.*

The preliminary note preceding the demonstration of the theorem states: *We have already seen that t either $= p - 1$ or $< p - 1$. It remains to show that in the latter case t will always be a factor of $p - 1$.* That remarkable property stimulates to go further back in section 3 thereby bringing to light details to the method of proof *more in harmony* with arithmetic:¹¹

45. THEOREM. *In any geometric progression $1, a, aa, a^3$ etc., outside of the first term 1, there is still another term a^t which is congruent to unity relative to the modulus p when p is prime relative to a , and the exponent t is $< p$.*

That theorem shows that the residues are generated from the leading 1 by repeated multiplication with the scale factor a of the progression — an exceptional and far-reaching phenomenon that has not been accentuated Dirichlet's lectures on number theory. Moreover section three with caption *Residues of Powers*¹² starts precisely with the general arithmetic truth providing an enlightening introduction to the first main theme of higher arithmetic which has been ignored in Dirichlet's lectures on number theory:

The residues of the terms of a geometric progressions which begins with unity constitute a periodic series¹³

Before coming to the first section supplying the basis of higher arithmetic these remarkable issues prompt to consider as well the preface of the *Disquisitiones Arithmeticae* with a splitting of arithmetic:¹⁴

¹⁰Gauss (1986, 30).

¹¹Gauss (1986, 29).

¹²Gauss (1986, 29–32).

¹³Marginal notes of Waterhouse's revised version of Clarke's English translation have been transformed to catchlines as put before in Schering's edition of the *Disquisitiones Arithmeticae* (Gauß, 1863). The original edition from 1801 does not provide that guidance but displays the fundamental arithmetic truth on page XIII of the table of contents.

¹⁴Gauss (1986, xvii).

However what is commonly called Arithmetic hardly extends beyond the art of enumerating and calculating (i.e. expressing numbers by suitable symbols, for example by a decimal representation, and carrying out arithmetic operations). As a result it seems proper to call what had been mentioned Elementary Arithmetic and to distinguish from it Higher Arithmetic which properly includes more general inquiries concerning integers. We consider in the present volume only Higher Arithmetic. — Included under the heading “Higher Arithmetic” are those topics which Euclid treated in Book VII ff. with the elegance and rigor customary among the ancients; but that restricts to the first commencements of this science.

2 Section one of the *Disquisitiones Arithmeticae*

CONGRUENT NUMBERS IN GENERAL¹⁵

Congruent numbers, modules, residues and nonresidues

1. If a number a divides the difference of the numbers b and c , b and c are said to be *congruent relative to a* ; if not, *noncongruent*. The number a is called the *modulus*. If the numbers b and c are congruent, each of them is called a *residue* of the other. If they are noncongruent they are called *nonresidues*.

The numbers involved must be positive or negative integers,* not fractions. For example, -9 and $+16$ are congruent modulo 5; -7 is a residue of $+15$ modulo 11, but a nonresidue modulo 3.

Since every number divides zero, it follows that we can regard any number as congruent to itself relative to any modulus.

The numerical presetting $-9 \equiv 16 \pmod{5}$ suggests to mark with the numbers -9 and 16 two points on the integer line and to consider with these the entire interval of residues modulo 5 located in between. Coloring congruent residues with the same color makes visible a geometric structure on the integer line resulting for the modulus 5 in consequence of the definition in article 1:

-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16

The sum of the residues 9 and 7 may be determined by starting from 9 and stepping forward 7 steps. Surprisingly however, the result 16 is congruent as well to any of $-9, -4, 1, 6$ and 11 and already that peculiarity indicates that higher arithmetic obeys rules which are considerably different from those governing elementary arithmetic as taught to school children.

¹⁵Gauss (1986, 1).

*Gauss's footnote: The modulus must obviously be taken absolutely, i.e. without sign.

2. Given a , all the residues modulo m are contained in the formula $a + km$ where k is any integer. The easier of the propositions that we state below follow at once from this, but with equal ease they can be proved directly, *as will be clear to the reader*.¹⁶

Henceforth we shall designate congruence by the symbol \equiv joining to it in parentheses the modulus when it is necessary to do so; e.g. $-7 \equiv 15 \pmod{11}$, $-16 \equiv 9 \pmod{5}$.^{**}

The formula $a + km$ from article 2 then implies that the pattern shown for the modulus 5 is periodic and repeats after five steps. Moreover, applying the symbol \equiv also to the sum $9 + 7 = 16$ and its different manifestations as $9 + 7 = 16 \equiv -9 \equiv -4 \equiv 1 \equiv 6 \equiv 11$ make visible a phenomenon typical for higher arithmetic which is inconceivable in its elementary shaping.

Article 3 then provides a straightforward means to assign to any given integer A its uniquely determined residue modulo m :¹⁷

3. THEOREM. *Let m successive integers*

$$a, a + 1, a + 2, \dots, a + m - 1$$

and another integer A be given; then one, and only one, of these integers will be congruent to A modulo m .

If $\frac{a-A}{m}$ is an integer then $a \equiv A$; if it is a fraction, let k be the next larger integer (or if it is negative, the next *smaller* integer not regarding sign). $A + km$ will fall between a and $a + m$ and will be the desired number. Evidently all the quotients

$$\frac{a-A}{m}, \quad \frac{a+1-A}{m}, \quad \frac{a+2-A}{m}, \quad \text{etc.}$$

lie between $k - 1$ and $k + 1$, so only one of them can be an integer.

Clearly that theorem makes obsolete Dirichlet's algorithm for the greatest common divisor. Moreover, an application of that theorem provides on the basis of *least residues* the first step to establish on the integer line an arithmetic of residues.

¹⁶Gauss (1986, 1). The emphasized text is missing in Waterhouse's revised edition and has been taken from Clarke's translation from 1966.

^{**}Gauss's footnote: We have adopted this symbol because of the analogy between equality and congruence. For the same reason Legendre, in the treatise which we shall often have occasion to cite, used the same sign for equality and congruence. To avoid ambiguity we have made a distinction.

¹⁷In order to avoid a line break inside the crucial datum of m successive residues the layout of the English translation (Gauss, 1986, 1–2) has been changed to Schering's reissue from 1863 (Gauß, 1863, 10) of the original latin text from 1801. Like changes of the English translation will occur also later *for the ease of order and summary* (Gauss, 1986, vi).

Least residues

4. Each number therefore will have a residue in the series $0, 1, 2, \dots, m-1$ and in the series $0, -1, -2, \dots, -(m-1)$. We will call these the *least residues*, and it is obvious that unless 0 is a residue, they always occur in pairs, one *positive* and one *negative*. If they are unequal in magnitude one will be $< m/2$; otherwise each will $= m/2$ disregarding sign. Thus each number has a residue which is not larger than half the modulus. It will be called the *absolutely least* residue.

For example, relative to the modulus 5, -13 has 2 as least positive residue. It is also the absolutely least residue, whereas -3 is the least negative residue. Relative to the modulus 7, $+5$ is its own least positive residue; -2 is the least negative residue and the absolutely least.

The definition of *least positive residues* and *least negative residues* permits to modify the residues of any two integers a and b and their sum $a + b$ in a way such that all three are contained in one and the same interval of length m . The following example shows for the modulus 5 how the sum $9 + 7 = 16$ may be determined in this manner: $9 + 7 \equiv 4 + 2 \equiv -1 + 2 = 1$.

Since for any given modulus m only the finitely many residues in the set $\{0, 1, \dots, m-1\}$ are concerned a general argument directed to all integers as in Dirichlet's improvements of elementary arithmetic provided by the *few arithmetic theorems, which indeed may be found in most text books, but which are of such fundamental importance for our science that a rigorous proof is necessary* are not necessary for higher arithmetic.

Any given arithmetic expression of residues may be treated for element by element and the independence of the value of a product of the order of the individual factors for example may be proved by the verifying finitely many cases. That renders unnecessary to involve schematic *complete induction* directed to *all* positive integers which therefore may seem *quite alien* to arithmetic.

Elementary propositions regarding congruences

5. Having established these concepts, let us establish the properties that follow from them.

Numbers that are congruent relative to a composite modulus are also congruent relative to any divisor of the modulus.

If many numbers are congruent to the same number relative to the same modulus, they are congruent to one another (relative to the same modulus).

This identity of moduli is to be understood also in what follows.

Congruent numbers have the same least residues, noncongruent numbers have different least residues.

6. *Given the numbers A, B, C , etc. and other numbers a, b, c , etc. congruent to each other relative to any modulus whatsoever, i.e. $A \equiv a, B \equiv b$, etc., then $A + B + C + \text{etc.} \equiv a + b + c + \text{etc.}$*

If $A \equiv a, B \equiv b$, then $A - B \equiv a - b$.

7. *If $A \equiv a$ then also $kA \equiv ka$.*

If k is a positive number, then this is only a particular case of the preceding article (art. 6) letting $A = B = C$ etc., $a = b = c$ etc. If k is negative, $-k$ will be positive. Thus $-kA \equiv -ka$ and so $kA \equiv ka$.

If $A \equiv a, B \equiv b$ then $AB \equiv ab$ because $AB \equiv Ab \equiv ba$.

8. *Given any numbers whatsoever A, B, C , etc. and other numbers a, b, c , etc. which are congruent to them, i.e. $A \equiv a, B \equiv b$, etc., the products of each will be congruent, i.e. ABC etc. $\equiv abc$ etc.*

From the preceding article $AB \equiv ab$ and for the same reason $ABC \equiv abc$ and any number of factors may be adjoined.

If all the numbers A, B, C , etc. and the corresponding a, b, c , etc. are assumed equal, then the following theorem holds: If $A \equiv a$ and k is a positive integer $A^k \equiv a^k$.

9. Let X be an algebraic function with undetermined x of the form

$$Ax^a + Bx^b + Cx^c + \text{etc.}$$

where A, B, C , etc. are any integers; a, b, c , etc. are nonnegative integers. Then if x is given values which are congruent relative to some modulus, the resulting values of the function X will also be congruent.

Let f, g be congruent values of x . Then from the preceding article $f^a \equiv g^a$ and $Af^a \equiv Ag^a$ and in the same way $Bf^b \equiv Bg^b$ etc. Thus

$$Af^a + Bf^b + Cf^c + \text{etc.} \equiv Ag^a + Bg^b + Cg^c + \text{etc.} \quad \text{Q.E.D.}$$

It is easy to understand how this theorem can be extended to functions of many undetermined variables.

Apparently Gauss's paradigm shift regarding mathematics with the transition from elementary to higher arithmetic¹⁸ has not yet been implemented in mathematics and the primordial importance of residues as autonomous

¹⁸Evenly Thomas Kuhn has not yet been able to point to that significant example in *The Structure of Scientific Revolutions* from 1962.

entities of arithmetic suitable to reveal unsuspected facets of the God given integers¹⁹ still looks forward to be recognized.

That fact manifests itself in a review from 1995 *On the arithmetic methods of mathematicians of the seventeenth century* in which Igor Rostislavovich Shafarevich wonders:²⁰

It remains a mystery why such simple objects as the integers require for their understanding practically the whole machinery which mathematicians are able to create. But this mystery is completely analogous to that enigmatic parallelism between mathematics and physics discussed by a number of men of science. In any case both phenomena are too universal to elucidate them using an improper stage of development. Apparently here we have to do with a cardinal phenomenon: the human way of thinking and the structure of the Cosmos are parallel to each other.

The obvious question if the mystery results all alone from the human way of thinking with regard to the integers is not taken into consideration.

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¹⁹According to Heinrich Weber (1892, 19) Leopold Kronecker (1823–1891) once proclaimed: “God made the integers; all else is the work of man”; see also *Namenverzeichnis* in Helmut Hasse (1950, 1, 467).

²⁰Luzin (1995, 1).